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Approximation of a Common Solution to Split Equilibrium and Fixed Point Problems for a Nonexpansive Semigroup a Modified Viscosity Method

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Abstract

The main objective of this paper is to suggest and analyze a new iterative algorithm, grounded in modified viscosity techniques, for approximating a common solution of a split equilibrium problem and a fixed-point problem associated with a non-expansive semigroup in Hilbert spaces. We prove that the sequence generated by the proposed scheme converges strongly to a point that simultaneously solves the split equilibrium problem and the corresponding fixed-point problem for the non-expansive semigroup. Furthermore, the findings presented here offer improvements, generalizations, and refinements of results found in literature.

Keywords: Split equilibrium problem; Fixed-point problem; Non-expansive semigroup; Iterative algorithms.

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1 Introduction

In the sequel, let H_1 and H_2 be real Hilbert spaces equipped with the standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $C \subset H_1$ and $Q \subset H_2$ are nonempty closed and convex sets.

We consider the following classical equilibrium problem (in short EP) due to Blum and Oettli [1]: Find $x \in C$ such that

$$F(x,y) \geq 0, \ \forall y \in C, \tag{1.1}$$

where $F: C \times C \to \mathbb{R}$ is a bifunction.

The EP(1.1) encompasses a wide range of mathematical models, including variational inequality problems, optimization problems, Nash equilibrium problems, saddle point problems, fixed point problems, and complementarity problems as particular cases. Hence, EP(1.1) serves as a unified framework for modeling and analyzing many problems that arise in science, engineering, optimization, economics, and related fields.

Over the past two decades, the equilibrium problem EP(1.1) has been widely generalized and thoroughly investigated in various directions because of its fundamental importance. For results concerning the existence of solutions for different extensions of EP(1.1), we refer the reader to [2,4,3,5] and the references cited therein. Numerous iterative algorithms have also been developed for solving different classes of equilibrium problems; see, for instance, [6, 7, 14, 8, 9, 12, 13, 10, 11] and references cited therein.

However, in many practical situations, equilibrium problems in a system are defined in different spaces. Motivated by this, we focus in this paper on a pair of equilibrium problems formulated in distinct spaces, known as the split equilibrium problem (SEP), which was introduced and studied by Kazmi and Rizvi [15].

Let $F_1: C \times C \to \mathbb{R}$ and $F_2: Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A: H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* SEP is to find $x^* \in C$ such that

$$F_1(x^*,x) \geq 0, \ \forall x \in C, \tag{1.2}$$

and such that

$$y^* = Ax^* \in Q \ \ solves \ F_2(y^*,y) \geq 0, \ \forall y \in Q. \tag{1.3}$$

EP(2.2) represents an equilibrium problem whose set of solutions is denoted by EP(F_1). The combined formulation (1.2)–(1.3) forms a pair of equilibrium problems that must be satisfied simultaneously. Specifically, one seeks a point x^* in the space H_1 that solves the equilibrium problem associated with (1.2), and whose image $y^* = Ax^*$, under a given bounded linear operator A solves another equilibrium problem defined in a second space H_2 as in (1.3). The solution set of the latter problem is denoted by EP(F_2). Accordingly, the set of solutions to the split equilibrium problem (1.2)–(1.3) is defined by $\Gamma = \{p \in \mathrm{E}P(F_1): Ap \in \mathrm{E}P(F_2)\}$.

The split equilibrium problem (1.2)–(1.3) can be viewed as a generalization of the multiple-sets split feasibility problem. Moreover, it encompasses, as a special case, the split variational inequality problem [16], which itself generalizes both split zero problems and split feasibility problems. For further details and related developments, we refer the reader to [20, 17, 21, 18, 19].

A family $S := \{T(s): 0 \leq s < \infty\}$ of mappings from C into itself is called *nonexpansive semigroup* on C if it satisfies the following conditions:

1. $T(0)x = x$ for all $x \in C$;
2. $T(s+t) = T(s)T(t)$ for all $s,t \geq 0$;
3. $\| T(s)x - T(s)y \| \leq \| x - y \|$ for all $x,y \in C$ and $s \geq 0$;

4. for all $x \in C$, $s \mapsto T(s)x$ is continuous.

The set of all the common fixed points of a family S is denoted by Fix(S), i.e.,

$$\mathrm{F}ix(S) := \{x \in C : T(s)x = x, 0 \le s < \infty\} = \bigcap_{0 \le s < \infty} \mathrm{F}ix(T(s)),$$

where $\mathrm{F}ix(T(s))$ is the set of fixed points of $T(s)$. It is well known that Fix(S) is closed and convex.

The *fixed point problem* (in short, FPP) for a nonexpansive semigroup S is:

$$\mathrm{F}ind\ x \in C\ such\ that\ x \in \mathrm{F}ix(S). \tag{1.4}$$

In 2006, Marino and Xu [24] considered the following implicit type iterative scheme for a nonexpansive mapping T:

$$x_t = t\gamma f(x_t) + (I - tP)Tx_t,$$

where f is a contraction mapping with constant α and $P: H_1 \to H_1$ is a strongly positive linear bounded operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Px, x\rangle \ge \bar{\gamma} \parallel x \parallel^2, \quad \forall x \in H_1,$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $t \in (0,1)$ and proved that the net (x_t) converges strongly to the unique solution of the variational inequality

$$\langle (P - \gamma f)z, x - z\rangle \ge 0, \ \forall x \in \mathrm{F}ix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{F}ix(T)} \frac{1}{2}\langle Px, x\rangle - h(x),$$

where h is the potential function for γf.

In 1997, Shimizu and Takahashi [25] introduced the following iterative method to prove the strong convergence theorem for FPP(1.4) in a real Hilbert space.

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\}$ is a sequence of positive real numbers which diverges to $+\infty$. Later, Chen and Song [26] introduced and studied the following iterative method to prove strong convergence theorem for FPP(1.4).

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where f is a contraction mapping. Recently, Plubtieng and Punpaeng [27] introduced and studied the following iterative method to prove a strong convergence theorem for FPP(1.4).

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ and $\{s_n\}$ is a positive real divergent sequence.

In 2010, Cianciaruso *et al.* [28] introduced and studied the following explicit iterative method for finding the common element of EP(1.1) and FPP(1.4)

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \forall y \in C; \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n P)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{s_n\}$ and $\{r_n\}$ are the sequences in $(0,1)$, under the certain appropriate conditions on control sequences. They obtain the strong convergence theorem.

In 2014, Kazmi and Rizvi [15] introduced and studied an Halpern type iterative method to approximate the common solution of split equilibrium problem , variatinal inequality problem and FPP for a nonexpansive mapping T on C. They proved that under some mild conditions, the sequences generated by the proposed iterative algorithms converges strongly to a common solution of SEP, VIP and

FPP for a nonexpansive mapping.

In 2018, Rizvi [13] introduced and studied an iterative method by combining Mann and Halpern iterative method to approximate the common solution of split mixed equilibrium problem and FPP for a nonexpansive mapping T. He proved that under some mild conditions, the sequences generated by the proposed iterative algorithms converges strongly to a common solution of split mixed equilibrium problem and FPP for a nonexpansive mapping.

In 2023, Rizvi [29] introduced and studied an iterative method based on viscosity iterative method to approximate the common solution of split variational inclusion problem and FPP for a nonexpansive mapping T. Further, He proved that under some mild conditions, the sequences generated by the proposed iterative algorithms converges strongly to a common solution of split variational inclusion problem and FPP for a nonexpansive mapping. For related work, see [30] and references quoted therein.

In 2024, Rizvi and Sikander [31] introduced and studied an iterative method based on hybrid iterative method to approximate the common solution of split generalized mixed equilibrium problem and FPP for a nonexpansive mapping T. Further, He proved that under some mild conditions, the sequences generated by the proposed iterative algorithms converges strongly to a common solution of split generalized mixed equilibrium problem and FPP for a nonexpansive mapping.

It is noteworthy that the analysis of weak and strong convergence theorems for finding a common solution to the SEP $(1.2)-(1.3)$ and the FPP (1.4) associated with a semigroup of nonexpansive mappings remains largely unexplored in the existing literature. Consequently, establishing strong convergence results for a common of SEP $(1.2)-(1.3)$ and FPP (1.4) for a nonexpansive semigroup in this setting constitutes a compelling, interesting and significant research problem.

Motivated by the work of Shimizu and Takahashi [25], Chen and Song [26], Plubtieng and Punpaeng [27], Cianciaruso *et al.* [28], and by the ongoing research in this direction, we suggest and analyze an iterative method based on modified viscosity iterative method to approximate a common solution of SEP(1.2)-(1.3) and FPP(1.4) for a nonexpansive semigroup in real Hilbert spaces. Furthermore, we prove that the sequences generated by the iterative scheme converges strongly to a common solution of SEP(1.2)-(1.3) and FPP(1.4). The results presented in this paper can be treated as a generalization of the work of Plubtieng [27] and Cianciaruso *et al.* [28].

2 Preliminaries

We recall some definitions and results which will be useful for the presentation of subsequent sections.

Definition 2.1 [32] *For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that*

$$\| x - P_C x \| \leq \| x - y \|, \ \forall y \in C.$$

P_C is called the *metric projection* of H_1 onto C.

Remark 2.1 *It is well known fact that P_C is nonexpansive mapping and satisfies the following properties:*

$$\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2, \ \forall x, y \in H_1. \tag{2.1}$$

Further, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.2}$$

and

$$\| x-y \|^2 \geq \| x-P_C x \|^2 + \| y-P_C x \|^2, \ \forall x \in H_1, y \in C. \quad (2.3)$$

Definition 2.2 [32] *In a real Hilbert space the following hold:*

1. $\| \lambda x+(1-\lambda)y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x-y \|^2$, (2.4)
for all $x, y \in H_1$ and $\lambda \in [0,1]$.
2. (Opial's condition) [33]. For any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality
$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|, \quad (2.5)$$
holds for every $y \in H_1$ with $y \neq x$.

Remark 2.2 *It is easy to observe that every nonexpansive operator $F: H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality*

$$\langle (x-F(x))-(y-F(y)), F(y)-F(x) \rangle \leq (1/2) \| (F(x)-x)-(F(y)-y \|^2, \quad (2.6)$$

therefore, we get, for all $(x, y) \in H_1 \times \mathrm{Fix}(F)$,

$$\langle x-F(x), y-F(x) \rangle \leq (1/2) \| F(x)-x \|^2, \quad (2.7)$$

see e.g., [[34], Theorem 2.1].

Lemma 2.1 [35] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{a_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 1$. Suppose $x_{n+1} = (1-a_n)y_n + a_n x_n$, for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\| y_{n+1}-y_n \| - \| x_{n+1}-x_n \|) \leq 0$. Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.*

Lemma 2.2 [32] *Assume that P is a strongly positive self-adjoint bounded linear operator on a Hilbert space H_1 with constant $\bar{\gamma} > 0$ and $0 < p \leq \| P \|^{-1}$. Then $\| I-\rho P \| \leq 1-\rho\bar{\gamma}$.*

Lemma 2.3 [25] *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H_1 and let $S := \{T(s): 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. For each $x \in C$ and $t > 0$. Then for any $0 \leq h < \infty$,*

$$\limsup_{t\to\infty} \sup_{x\in C} \| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right) \| = 0.$$

Assumption 2.1 [1] *Let $F: C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i). $F(x,x) = 0, \ \forall x \in C$;
(ii) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0, \ \forall x \in C$;
(iii) For each $x, y, z \in C$, $\limsup_{t\to 0} F(tz+(1-t)x, y) \leq F(x,y)$;
(iv) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous.

Lemma 2.4 [14] *Assume that $F: C \times C \to \mathbb{R}$ satisfying Assumption* 3.1. *For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^F: H_1 \to C$ as follows:*

$$T_r^F x = \{z \in C: F(z,y) + \frac{1}{r}\langle y-z, z-x \rangle \geq 0, \ \forall y \in C\}.$$

Then the following hold:

(i) T_r^F is single-valued;
(ii) T_r^F is firmly nonexpansive, i.e.,
$$\| T_r^F x - T_r^F y \|^2 \leq \langle T_r^F x - T_r^F y, x-y \rangle, \ \forall x, y \in H_1;$$
(iii) $\mathrm{Fix}(T_r^F) = Sol(EP(1.2))$;
(iv) Sol(EP(1.2)) is closed and convex.

Moreover, assume that $G: Q \times Q \to \mathbb{R}$ satisfying Assumption 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^G: H_2 \to Q$ as follows:

$$T_s^G w = \{d \in Q: \exists z \in T_r^F x \text{ such that } Az = d \text{ and } G(d,y) + \frac{1}{s}\langle d - e, e - w\rangle \geq 0, \forall e \in Q\}.$$

Then, we can easily observe that T_s^G is single-valued and firmly nonexpansive, Sol(EP(1.3)) is closed and convex and $\text{Fix}(T_s^G) = Sol(EP(1.3))$.

Lemma 2.5 [28] *Let $F: C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption* 3.1 *hold and let T_r^F be defined as in Lemma* 2.4 *for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then:*

$$\| T_{r_2}^F y - T_{r_1}^F x \| \leq \| y - x \| + \left|\frac{r_2 - r_1}{r_2}\right| \| T_{r_2}^F y - y \|.$$

Lemma 2.6 [36] *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, then for all $x, y \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 - \alpha\beta \| x - y \|^2 - \alpha\gamma \| x - z \|^2 - \beta\gamma \| y - z \|^2.$$

Lemma 2.7 [37] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

1. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
2. $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main Result

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A: H_1 \to H_2$ be a bounded linear self adjoint operator. Assume that $F_1: C \times C \to \mathbb{R}$ and $F_2: Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in first argument. Let $S = \{T(s): 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $f: C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and P be a strongly positive bounded linear operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n P)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{3.1}$$

where $r_n \subset (0, \infty)$ and $\delta \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

1. $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; and $\sum |\alpha_{n+1} - \alpha_n| < \infty$;
2. $\liminf r_n > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;
3. $\lim_{n\to\infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \text{F}ix(S) \cap \Gamma$, where $z = P_{\text{F}ix(S)\cap\Gamma}(I - P + \gamma f)z$.

Proof. We note that from condition (i), we may assume without loss of generality that $\alpha_n \leq (1-\beta_n) \parallel P \parallel^{-1}$ for all n. From Lemma 2.2, we know that if $0 < \rho \leq \parallel P \parallel^{-1}$, then $\parallel I - \rho P \parallel \leq 1 - \rho\bar{\gamma}$. We will assume that $\parallel I - P \parallel \leq 1 - \bar{\gamma}$.

Since P is positive linear bounded self-adjoint operator on H_1, then

$$\parallel P \parallel = \sup\{|\langle Pu, u\rangle| : u \in H_1, \parallel u \parallel = 1\}.$$

Observe that

$$\begin{aligned}\langle (I-\alpha_n P)u, u\rangle &= 1 - \alpha_n\langle Pu, u\rangle \\ &\geq 1 - \alpha_n \parallel P \parallel \\ &\geq 0,\end{aligned}$$

which implies that $(1-\alpha_n P)$ is positive. It follows that

$$\begin{aligned}\parallel (I-\alpha_n P \parallel &= \sup\{\langle((1-\alpha_n P)u, u\rangle : u \in H_1, \parallel u \parallel = 1\} \\ &= \sup\{1 - \alpha_n\langle Pu, u\rangle : u \in H_1, \parallel u \parallel = 1\} \\ &\leq 1 - \alpha_n\bar{\gamma}.\end{aligned}$$

Let $q = P_{\mathrm{Fix}(S)\cap\Gamma}$. Since f is a contraction mapping with constant $\alpha \in (0,1)$. It follows that

$$\begin{aligned}\parallel q(I-P+\gamma f)(x) - q(I-P+\gamma f)(y) \parallel &\leq \parallel (I-P+\gamma f)(x) - (I-P+\gamma f)(x) \parallel \\ &\leq \parallel I-P \parallel\parallel x-y \parallel + \gamma \parallel f(x) - f(y) \parallel \\ &\leq (1-\bar{\gamma}) \parallel x-y \parallel + \gamma\alpha \parallel x-y \parallel \\ &\leq (1-(\bar{\gamma}-\gamma\alpha)) \parallel x-y \parallel,\end{aligned}$$

for all $x, y \in H_1$. Therefore the mapping $q(I-P+\gamma f)$ is a contraction mapping from H_1 into itself. It follows from Banach contraction principle that there exists an element $z \in H_1$ such that $z = q(I-P+\gamma f)z = P_{\mathrm{Fix}(S)\cap\Gamma}(I-P+\gamma f)(z)$.

Let $p \in Fix(S) \cap \Gamma$, i.e., $p \in \Gamma$, we have $p = T_{r_n}^{F_1} p$ and $Ap = T_{r_n}^{F_2}(Ap)$.

We estimate

$$\begin{aligned}\parallel u_n - p \parallel^2 &= \parallel T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - p \parallel^2 \\ &= \parallel T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1} p \parallel^2 \\ &\leq \parallel x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p \parallel^2 \\ &\leq \parallel x_n - p \parallel^2 + \delta^2 \parallel A^*(T_{r_n}^{F_2} - I)Ax_n \parallel^2 \\ &+ 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \qquad (3.2)\end{aligned}$$

Thus, we have

$$\begin{aligned}\parallel u_n - p \parallel^2 &\leq \parallel x_n - p \parallel^2 + \delta^2\langle(T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &+ 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \qquad (3.3)\end{aligned}$$

Now, we have

$$\begin{aligned}&\delta^2\langle(T_{r_n}^{(F_2,h_2)} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &\leq L\delta^2\langle(T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= L\delta^2 \parallel (T_{r_n}^{F_2} - I)Ax_n \parallel^2. \qquad (3.4)\end{aligned}$$

$Denoting\ \mathrm{F} = 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle$ and using (2.7), we have

$$\begin{aligned}\Lambda &= 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\delta\langle A(x_n - p), (T_{r_n}^{F_2} - I)Ax_n\rangle\end{aligned}$$

$$= 2\delta\langle A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n - (T_r^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle$$
$$= 2\delta\{\langle T_{r_n}^{F_2}Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n\rangle - \| (T_{r_n}^{F_2} - I)Ax_n \|^2\}$$
$$\leq 2\delta\{\tfrac{1}{2} \| (T_{r_n}^{F_2} - I)Ax_n \|^2 - \| (T_{r_n}^{F_2} - I)Ax_n \|^2\}$$
$$\leq -\delta \| (T_{r_n}^{F_2} - I)Ax_n \|^2. \quad (3.5)$$

Using (3.3), (3.4) and (3.5), we obtain

$$\| u_n - p \|^2 \leq \| x_n - p \|^2 + \delta(L\delta - 1) \| (T_{r_n}^{F_2} - I)Ax_n \|^2. \quad (3.6)$$

Since, $\delta \in (0, \frac{1}{L})$, we obtain

$$\| u_n - p \|^2 \leq \| x_n - p \|^2. \quad (3.7)$$

Now setting $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$ and since $p \in Fix(S) \cap \Gamma$, we obtain

$$\| t_n - p \| = \| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p \|$$
$$\leq \frac{1}{s_n}\int_0^{s_n} \| T(s)u_n - T(s)p \| ds$$
$$\leq \| u_n - p \|$$
$$\leq \| x_n - p \|. \quad (3.8)$$

Further, we estimate

$$\| x_{n+1} - p \| = \| \alpha_n\gamma f(x_n) + (I - \alpha_n P)t_n - p \|$$
$$= \| \alpha_n(\gamma f(x_n) - Pp) + (I - \alpha_n P)(t_n - p) \|$$
$$\leq \alpha_n \| \gamma f(x_n) - Pp \| + (1 - \alpha_n\bar{\gamma}) \| t_n - p \|$$
$$\leq \alpha_n\gamma \| f(x_n) - f(p) \| + \alpha_n \| \gamma f(p) - Pp \| + (1 - \alpha_n\bar{\gamma}) \| x_n - p \|$$
$$\leq \alpha_n\gamma\alpha \| x_n - p \| + \alpha_n \| \gamma f(p) - Pp \| + (1 - \alpha_n\bar{\gamma}) \| x_n - p \|$$
$$= (1 - (\bar{\gamma} - \gamma\alpha)\alpha_n) \| x_n - p \| + \alpha_n \| \gamma f(p) - Pp \|$$
$$\leq max\{\| x_n - p \|, \frac{1}{\bar{\gamma} - \gamma\alpha} \| \gamma f(p) - Pp \|\}, \quad n \geq 0$$
$$\vdots$$
$$\leq max\{\| x_0 - p \|, \frac{1}{\bar{\gamma} - \gamma\alpha} \| \gamma f(p) - Pp \|\}. \quad (3.9)$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Next, we estimate

$$\| t_{n+1} - t_n \| = \| \frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1} ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \|$$
$$= \| \frac{1}{s_{n+1}}\int_0^{s_{n+1}} [T(s)u_{n+1} - T(s)u_n] ds + (\frac{1}{s_{n+1}} - \frac{1}{s_n}) \times$$
$$\int_0^{s_n} T(s)u_n ds + \frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}} T(s)u_n ds \|$$
$$= \| \frac{1}{s_{n+1}}\int_0^{s_{n+1}} [T(s)u_{n+1} - T(s)u_n] ds + (\frac{1}{s_{n+1}} - \frac{1}{s_n}) \times$$
$$\int_0^{s_n} [T(s)u_n - T(s)p] ds + \frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}} [T(s)u_n - T(s)p] ds \|$$
$$\leq \| u_{n+1} - u_n \| + \frac{|s_{n+1} - s_n|s_n}{(s_{n+1})s_n} \| u_n - p \| + \frac{|s_{n+1} - s_n|}{s_{n+1}} \| u_n - p \|$$
$$\leq \| u_{n+1} - u_n \| + 2\frac{|s_{n+1} - s_n|}{s_{n+1}} \| u_n - p \|. \quad (3.10)$$

Since $u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)$ and $u_{n+1} = T_{r_{n+1}}^{F_1}(x_{n+1} + \delta A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1})$.

It follows from Lemma 2.5 that

$$\begin{aligned}
&\| u_{n+1} - u_n \| \leq \| x_{n+1} - x_n + \delta[A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1} - A^*(T_{r_n}^{F_2} - I)Ax_n] \| \\
&+ \left|1 - \frac{r_n}{r_{n+1}}\right| \| T_{r_{n+1}}^{F_1}(x_{n+1} + \delta A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1}) \\
&-(x_{n+1} + \delta A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1}) \| \\
&\leq \| x_{n+1} - x_n + \delta A^*A(x_{n+1} - x_n) \| + \delta \| A \| \| T_{r_{n+1}}^{F_2} Ax_{n+1} - T_{r_n}^{F_2,} Ax_n \| \\
&+\delta_n, \\
&\leq \{\| x_{n+1} - x_n \|^2 - 2\delta \| Ax_{n+1} - Ax_n \|^2 + \delta^2 \| A \|^4 \| x_{n+1} - x_n \|^2\}^{\frac{1}{2}} \\
&+\delta \| A \| \{\| Ax_{n+1} - Ax_n \| + \left|1 - \frac{r_n}{r_{n+1}}\right| \| T_{r_{n+1}}^{F_2)} Ax_{n+1} - Ax_{n+1} \|\} + \delta_n \\
&\leq (1 - 2\delta \| A \|^2 + \delta^2 \| A \|^4)^{\frac{1}{2}} \| x_{n+1} - x_n \| + \delta \| A \|^2 \| x_{n+1} - x_n \| \\
&+\delta \| A \| \sigma_n + \delta_n \\
&= (1 - \delta \| A \|^2) \| x_{n+1} - x_n \| + \delta \| A \|^2 \| x_{n+1} - x_n \| + \delta \| A \| \sigma_n + \delta_n, \\
&= \| x_{n+1} - x_n \| + \delta \| A \| \sigma_n + \delta_n, \qquad (3.11)
\end{aligned}$$

where

$$\sigma_n = \left|1 - \frac{r_n}{r_{n+1}}\right| \| T_{r_{n+1}}^{F_2} Ax_{n+1} - Ax_{n+1} \|$$

and

$$\delta_n = \left|1 - \frac{r_n}{r_{n+1}}\right| \| T_{r_{n+1}}^{F_1}(x_{n+1} + \delta A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1}) - (x_{n+1} + \delta A^*(T_{r_{n+1}}^{F_2} - I)Ax_{n+1}) \|.$$

Using (3.10) and (3.11), we have

$$\| t_{n+1} - t_n \| \leq \| x_{n+1} - x_n \| + \delta \| A \| \sigma_n + \delta_n + 2\frac{|s_{n+1} - s_n|}{s_{n+1}} \| u_n - p \|. \qquad (3.12)$$

Next, we show that the sequence $\{x_n\}$ is asymptotically regular

$$\begin{aligned}
\| x_{n+1} - x_n \| &= \| \alpha_n \gamma f(x_n) + (1 - \alpha_n P)t_n - [\alpha_{n-1}\gamma f(x_{n-1}) + (1 - \alpha_{n-1}P)t_{n-1}] \| \\
&= \| \alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1}) + \alpha_n \gamma f(x_{n-1}) - \alpha_{n-1}\gamma f(x_{n-1}) \\
&+(1 - \alpha_n P)t_n - (1 - \alpha_n P)t_{n-1} + (1 - \alpha_n` P)t_{n-1} - (1 - \alpha_{n-1}P)t_{n-1} \| \\
&\leq \alpha_n \gamma \alpha \| x_n - x_{n-1} \| + \gamma|\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + \| I - \alpha_n)P \| \| t_n - t_{n-1} \| \\
&+|\alpha_n - \alpha_{n-1}| \| Pt_{n-1} \|, \\
&\leq \alpha_n \gamma \alpha \| x_n - x_{n-1} \| \\
&+(1 + \gamma)|\alpha_n - \alpha_{n-1}|K_1 + (1 - \alpha_n \bar{\gamma}) \| t_n - t_{n-1} \| \qquad (3.13)
\end{aligned}$$

where $K_1 = \sup\{\| f(x_n) \| + \| Pt_n \| : n \in N\} < \infty$. It follows from (3.12) and (3.13) that

$$\begin{aligned}
\| x_{n+1} - x_n \| &\leq \alpha_n \gamma \alpha \| x_n - x_{n-1} \| + (1 + \gamma)|\alpha_n - \alpha_{n-1}|K_1 + (1 - \alpha_n \bar{\gamma})[\| x_n - x_{n-1} \| \\
&+\delta \| A \| \sigma_n + \delta_n + 2\frac{|s_n - s_{n-1}|}{s_n} \| u_{n-1} - p \|], \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \| x_n - x_{n-1} \| + (1 + \gamma)|\alpha_n - \alpha_{n-1}|K_1 \\
&+(1 - \alpha_n \bar{\gamma}) \left[\delta \| A \| \sigma_n + \delta_n + 2\frac{|s_n - s_{n-1}|}{s_n} \| u_{n-1} - p \|\right] \\
&\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \| x_n - x_{n-1} \| + (1 + \gamma)|\alpha_n - \alpha_{n-1}|K_1 \\
&+\delta \| A \| \sigma_n + \delta_n + 2\frac{|s_n - s_{n-1}|}{s_n} \| u_{n-1} - p \| \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \| x_n - x_{n-1} \| + (1 + \gamma)|\alpha_n - \alpha_{n-1}|K_1 \\
&+\delta \| A \| \left|1 - \frac{r_n}{r_{n+1}}\right| K_2 + \left|1 - \frac{r_n}{r_{n+1}}\right| K_3 + 2\frac{|s_n - s_{n-1}|}{s_n} K_4, \qquad (3.14)
\end{aligned}$$

where $K_2 = \sup\{\| (T_{r_n}^{F_2} - I)Ax_n \| : n \in N\}, K_3 = \sup\{\| u_n - x_n + \delta A^* K_2 \| : n \in N\}$ and $K_4 =$

$\sup\{\| u_n - p \|: n \in N\}$. Setting $K = \max\{K_1, K_2, K_3, K_4\}$. It follows from (3.14) that

$$\| x_{n+1} - x_n \| \leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \| x_n - x_{n-1} \| + K\left[\left|1 - \frac{r_n}{r_{n+1}}\right| + 2\frac{|s_n - s_{n-1}|}{s_n} + (1+\gamma)|\alpha_n - \alpha_{n-1}|\right] \quad (3.15)$$

Hence by Lemma 2.7 and (i)-(iii), we have

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \quad (3.16)$$

Now, $x_{n+1} - x_n = \alpha_n\gamma f(x_n) + (I - \alpha_n P)t_n - x_n$
$= \alpha_n(\gamma f(x_n) - x_n) + (I - \alpha_n P)(t_n - x_n)$.

Since $\| x_{n+1} - x_n \| \to 0$ as $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \| t_n - x_n \| = 0. \quad (3.17)$$

Next, we have

$$\begin{aligned}
&\| T(s)x_n - x_n \| = \| T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds + T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&-\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds + \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n \| \\
&\leq \| T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \| \\
&+ \| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \| \\
&+ \| \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - x_n \| \\
&\leq \| x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \| \\
&+ \| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \| \\
&+ \| \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - x_n \| \\
&\leq 2 \| x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \| \\
&+ \| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \|. \quad (3.18)
\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ is bounded. Let $K := \{w \in C: \| w - p \| \leq \| x_0 - p \|, \frac{1}{\bar{\gamma} - \gamma\alpha} \| \gamma f(p) - Pp \|\}$, then K is nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $0 \leq s < \infty$ and contains $\{x_n\}$. So without loss of generality, we may assume that $S := \{T(s): 0 \leq s < \infty\}$ is nonexpansive semigroup on K. By Lemma 2.3, we have

$$\lim_{n\to\infty} \| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \| = 0. \quad (3.19)$$

Using (3.10)-(3.12), we obtain

$$\lim_{n\to\infty} \| T(s)x_n - x_n \| = 0. \quad (3.20)$$

It follows from (3.6) and Lemma 2.6 (i) that

$$\begin{aligned}
&\| x_{n+1} - p \|^2 = \| \alpha_n\gamma f(x_n) + (I - \alpha_n P)t_n - p \|^2 \\
&= \| \alpha_n(\gamma f(x_n) - Pp) + (I - \alpha_n P)(t_n - p) \|^2 \\
&\leq \| (I - \alpha_n P)(t_n - p) \|^2 + 2\alpha_n\langle \gamma f(x_n) - Pp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\bar{\gamma})^2 \| t_n - p \|^2 + 2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|
\end{aligned}$$

$$= (I - \alpha_n\bar{\gamma})^2 \| u_n - p \|^2 + 2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \| \quad (3.21)$$
$$\leq (1 - \alpha_n\bar{\gamma})^2[\| x_n - p \|^2 + \delta(L\delta - 1) \| (T_{r_n}^{F_2} - I)Ax_n \|^2]$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|$$
$$\leq [1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2] \| x_n - p \|^2 + (1 - \alpha_n\bar{\gamma})^2\delta(L\delta - 1) \| (T_{r_n}^{F_2} - I)Ax_n \|^2$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|$$
$$\leq \| x_n - p \|^2 + \alpha_n\bar{\gamma}^2 \| x_n - p \|^2 + (1 - \alpha_n\bar{\gamma})^2\delta(L\delta - 1) \| (T_{r_n}^{F_2} - I)Ax_n \|^2$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|.$$

Therefore,

$$(1 - \alpha_n\bar{\gamma})^2\delta(1 - L\delta) \| (T_{r_n}^{F_2} - I)Ax_n \|^2$$
$$\leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + \alpha_n\bar{\gamma}^2 \| x_n - p \|^2$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|$$
$$\leq (\| x_n - p \| + \| x_{n+1} - p \|) \| x_n - x_{n+1} \| + \alpha_n\bar{\gamma}^2 \| x_n - p \|^2$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|.$$

Since $\delta(1 - L\delta) > 0$, $\alpha_n \to 0$, $\| x_n - t_n \| \to 0$ and $\| x_{n+1} - x_n \| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \| (T_{r_n}^{F_2} - I)Ax_n \| = 0. \quad (3.22)$$

Next, we show that $\| x_n - u_n \| \to 0$ as $n \to \infty$. Since $p \in \mathrm{F}ix(S) \cap \Gamma$, we obtain

$$\| u_n - p \|^2 = \| T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - p \|^2$$
$$= \| T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}p \|^2$$
$$\leq \langle u_n - p, x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p \rangle$$
$$= \frac{1}{2}\{\| u_n - p \|^2 + \| x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p \|^2$$
$$- \| (u_n - p) - [x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p] \|^2\}$$
$$= \frac{1}{2}\{\| u_n - p \|^2 + \| x_n - p \|^2 - \| u_n - x_n - \delta A^*(T_{r_n}^{F_2} - I)Ax_n \|^2\}$$
$$= \frac{1}{2}\{\| u_n - p \|^2 + \| x_n - p \|^2 - [\| u_n - x_n \|^2$$
$$+\delta^2 \| A^*(T_{r_n}^{F_2} - I)Ax_n \|^2 - 2\delta\langle u_n - x_n, A^*(T_{r_n}^{F_2} - I)Ax_n \rangle]\}.$$

Hence, we obtain

$$\| u_n - p \|^2 \leq \| x_n - p \|^2 - \| u_n - x_n \|^2 + 2\delta \| A(u_n - x_n) \| \| T_{r_n}^{F_2} - I)Ax_n \|.$$

It follows from (3.21) and (3.22) that

$$\| x_{n+1} - p \|^2 \leq (1 - \alpha_n\bar{\gamma})^2[\| x_n - p \|^2 - \| u_n - x_n \|^2$$
$$+2\delta \| A(u_n - x_n) \| \| T_{r_n}^{F_2} - I)Ax_n \|]$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|$$
$$\leq [1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2] \| x_n - p \|^2 - (1 - \alpha_n\bar{\gamma})^2 \| u_n - x_n \|^2$$
$$+2(1 - \alpha_n\bar{\gamma})^2\delta \| A(u_n - x_n) \| \| T_{r_n}^{F_2} - I)Ax_n \|$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|$$
$$\leq \| x_n - p \|^2 + \alpha_n\bar{\gamma}^2 \| x_n - p \|^2 - (1 - \alpha_n\bar{\gamma})^2 \| u_n - x_n \|^2$$
$$+2(1 - \alpha_n\bar{\gamma})^2\delta \| A(u_n - x_n) \| \| T_{r_n}^{F_2} - I)Ax_n \|$$
$$+2\alpha_n \| \gamma f(x_n) - Pp \| \| x_{n+1} - p \|.$$

Therefore,

$$\begin{aligned}(1-\alpha_n\bar{\gamma})^2 \parallel u_n - x_n \parallel^2 &\leq \parallel x_n - p \parallel^2 - \parallel x_{n+1} - p \parallel^2 \\ &+2(1-\alpha_n\bar{\gamma})^2\delta \parallel A(u_n - x_n) \parallel \parallel T_{r_n}^{F_2} - I)Ax_n \parallel \\ &+\alpha_n\bar{\gamma}^2 \parallel x_n - p \parallel^2 \\ &+2\alpha_n \parallel \gamma f(x_n) - Pp \parallel \parallel x_{n+1} - p \parallel \\ &\leq (\parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel) \parallel x_n - x_{n+1} \parallel \\ &+2(1-\alpha_n\bar{\gamma})^2\delta \parallel A(u_n - x_n) \parallel \parallel T_{r_n}^{F_2} - I)Ax_n \parallel \\ &+\alpha_n\bar{\gamma}^2 \parallel x_n - p \parallel^2 + 2(1-\alpha_n\bar{\gamma})\beta_n \parallel u_n - p \parallel \parallel x_n - t_n \parallel \\ &+2\alpha_n \parallel \gamma f(x_n) - Pp \parallel \parallel x_{n+1} - p \parallel.\end{aligned}$$

Since $\alpha_n \to 0$, $\parallel x_n - t_n \parallel \to 0$, $\parallel T_{r_n}^{F_2} - I)Ax_n \parallel \to 0$ and $\parallel x_{n+1} - x_n \parallel \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \parallel u_n - x_n \parallel = 0. \tag{3.23}$$

Since, we can write

$$\begin{aligned}\parallel T(s)t_n - x_n \parallel &\leq \parallel T(s)t_n - T(s)x_n \parallel + \parallel T(s)x_n - x_n \parallel \\ &\leq \parallel t_n - x_n \parallel + \parallel T(s)x_n - x_n \parallel \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Also, we have

$$\begin{aligned}\parallel T(s)t_n - t_n \parallel &\leq \parallel T(s)t_n - T(s)x_n \parallel + \parallel T(s)x_n - x_n \parallel + \parallel x_n - t_n \parallel \\ &\leq \parallel t_n - x_n \parallel + \parallel T(s)x_n - x_n \parallel + \parallel x_n - t_n \parallel \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Next, we show that $\limsup_{n\to\infty}\langle (P-\gamma f)z, x_n - z\rangle \leq 0$, where $z = P_{\mathrm{F}ix(S)\cap\Gamma}(I - P + \gamma f)z$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\} \subseteq K$ such that

$$\limsup_{n\to\infty}\langle (P-\gamma f)z, t_n - z\rangle = \lim_{i\to\infty}\langle (P-\gamma f)z, t_{n_i} - z\rangle.$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup w$.

Now, we prove that $w \in \mathrm{F}ix(S) \cap \Gamma$. Let us first show that $w \in \mathrm{F}ix(S)$. Assume that $w \notin \mathrm{F}ix(S)$. Since $t_{n_i} \rightharpoonup w$ and $T(s)w \neq w$. From Opial's condition (2.5), we have

$$\begin{aligned}\liminf_{i\to\infty} \parallel t_{n_i} - w \parallel &< \liminf_{i\to\infty} \parallel t_{n_i} - T(s)w \parallel \\ &\leq \liminf_{i\to\infty}\{\parallel t_{n_i} - T(s)t_{n_i} \parallel + \parallel T(s)t_{n_i} - T(s)w \parallel\} \\ &\leq \liminf_{i\to\infty} \parallel t_{n_i} - w \parallel,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{F}ix(S)$.

Next, we show that $w \in \mathrm{E}P(F_1)$. Since $u_n = T_{r_n}^{F_1}x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C.$$

It follows from monotonicity of F_1 that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n}\right\rangle \geq F_1(y, u_{n_i}).$$

Since $\parallel u_n - x_n \parallel \to 0$ we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$. It follows by Assumption 2.1 (iv) that $0 \geq F_1(y, w), \forall w \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$, $w \in C$,

we get $y_t \in C$ and hence $F_1(y_t, w) \leq 0$. So from Assumption 2.1 (i) and (iv) we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, w)$$
$$\leq tF_1(y_t, y).$$

Therefore $0 \leq F_1(y_t, y)$. From Assumption 2.1 (iii), we have $0 \leq F_1(w, y)$. This implies that $w \in \mathrm{E}P(F_1)$.

Next, we show that $w \in \mathrm{E}P(F_2)$. Since $\| u_n - x_n \| \to 0, u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w$ and since A is a bounded linear operator so that $Ax_{n_k} \rightharpoonup Aw$.

Now setting $v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2} Ax_{n_k}$. It follows that from (3.22) that $\lim_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{F_2} Ax_{n_k}$.

Therefore from Lemma 2.4, we have

$$F_2(Ax_{n_k} - v_{n_k}, z) + \frac{1}{r_{n_k}} \langle z - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k} \rangle \geq 0, \forall z \in Q.$$

Since F_2 is upper semicontinuous in first argument, taking $\liminf$ to above inequality as $k \to \infty$ and using condition (iv), we obtain

$$F_2(Aw, z) \geq 0, \forall z \in Q,$$

which means that $Aw \in \mathrm{E}P(F_2)$ and hence $w \in \Gamma$.

Next, we claim that $\limsup_{n\to\infty} \langle f(z) - z, x_n - z \rangle \leq 0$, where $z = P_{\mathrm{F}ix(S)\cap\Gamma}(I - P + \gamma f)z$. Now from (2.2), we have

$$\limsup_{n\to\infty} \langle (P - \gamma f)z - z, x_n - z \rangle = \limsup_{n\to\infty} \langle (P - \gamma f)z - z, t_n - z \rangle$$
$$\leq \limsup_{i\to\infty} \langle (P - \gamma f)z - z, t_{n_i} - z \rangle$$
$$= \langle (P - \gamma f)z - z, w - z \rangle$$
$$\leq 0. \qquad (3.24)$$

Finally, we show that $x_n \to z$.

$$\| x_{n+1} - z \|^2 = \| \alpha_n \gamma f(x_n) + [(I - \alpha_n P)t_n - z \|^2$$
$$= \| \alpha_n(\gamma f(x_n) - Pz) + (I - \alpha_n P)(t_n - z) \|^2$$
$$\leq \| (I - \alpha_n P)(t_n - z) \|^2 + 2\alpha_n \langle \gamma f(x_n) - Pz, x_{n+1} - z \rangle$$
$$\leq (1 - \alpha_n \bar{\gamma})^2 \| t_n - z \|^2 + 2\alpha_n \gamma \langle f(x_n) - f(z), x_{n+1} - z \rangle$$
$$+ 2\alpha_n \langle \gamma f(z) - Pz, x_{n+1} - z \rangle$$
$$\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - z \|^2 + 2\alpha_n \gamma \alpha \| x_n - z \| \| x_{n+1} - z \|$$
$$+ 2\alpha_n \langle \gamma f(z) - Pz, x_{n+1} - z \rangle$$
$$\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - z \|^2 + \alpha_n \gamma \alpha \{ \| x_n - z \|^2 + \| x_{n+1} - z \|^2 \}$$
$$+ 2\alpha_n \langle \gamma f(z) - Pz, x_{n+1} - z \rangle$$
$$\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - z \|^2 + \alpha_n \gamma \alpha \| x_n - z \|^2 + \gamma \alpha \| x_{n+1} - z \|^2$$
$$+ 2\alpha_n \langle \gamma f(z) - Pz, x_{n+1} - z \rangle.$$

This implies that

$$\| x_{n+1} - z \|^2 \leq \frac{1 - 2\alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \gamma \alpha} \| x_n - z \|^2$$
$$+ \frac{2\alpha_n}{1 - \gamma \alpha} \langle \gamma f(z) - Pz, x_{n+1} - z \rangle$$
$$= \left[1 - \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha}\right] \| x_n - z \|^2 + \frac{(\alpha_n \bar{\gamma})^2}{1 - \gamma \alpha} \| x_n - z \|^2$$

$$+\frac{2\alpha_n}{1-\gamma\alpha}\langle \gamma f(z)-Pz, x_{n+1}-z\rangle$$
$$\leq [1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}] \parallel x_n-z \parallel^2+\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}$$
$$\times \{\frac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma}-\gamma\alpha)}+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle \gamma f(z)-Pz, x_{n+1}-z\rangle\}$$
$$=(1-\delta_n) \parallel x_n-z \parallel^2+\delta_n\sigma_n, \quad (3.25)$$

where $M:=\sup\{\parallel x_n-z \parallel^2: n\geq 1\}$, $\delta_n=\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}$ and $\sigma_n=\frac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma}-\gamma\alpha)}+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle \gamma f(z)-Pz, x_{n+1}-z\rangle$. Since $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, it is easy to see that $\lim_{n\to\infty}\delta_n=0$, $\sum_{n=0}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\sigma_n\leq 0$. Hence from (3.24), (3.25) and Lemma 2.7, we deduce that $x_n\to z$. This completes the proof.

We have the following consequences of Theorem 3.1.

Corollary 3.1 [28] *Let H be a real Hilbert space and let $C\subseteq H$ be nonempty closed convex subset. Assume that $F: C\times C\to\mathbb{R}$ be a bifunction satisfying Assumption* 2.1 *for F only. Let $S=\{T(s): 0\leq s<\infty\}$ be a nonexpansive semigroup on C such that $Fix(S)\cap EP(F)=\emptyset$. Let $f: C\to C$ be a contraction mapping with constant $\alpha\in(0,1)$ and P be a strongly positive bounded linear operator on H with constant $\bar{\gamma}>0$ such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}<\gamma+\frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0\in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n=T_{r_n}^F x_n; \\ x_{n+1}=\alpha_n\gamma f(x_n)+(1-\alpha_n P)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{cases}$$

where $r_n\subset(0,\infty)$ and $\{\alpha_n\}$ is a sequences in $(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $\liminf_{n\to\infty} r_n>0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$;

(iii) $\lim_{n\to\infty}\frac{|s_{n+1}-s_n|}{s_{n+1}}=0$.

Then the sequence $\{x_n\}$ converges strongly to $z\in \mathrm{F}ix(S)\cap \mathrm{E}P(F)$, where $z=P_{\mathrm{F}ix(S)\cap \mathrm{E}P(F)}(I-P+\gamma f)z$.

Proof. Taking $F_1=F_2=F$, $H_1=H_2=H$ and $A=0$ in Theorem 3.1 then the conclusion of Corollary 3.1 is obtained.

Corollary 3.2 *Let H be a real Hilbert space and let $C\subseteq H$ be a nonempty closed convex subset. Let $S=\{T(s): 0\leq s<\infty\}$ be a nonexpansive semigroup on C such that $Fix(S)\neq\emptyset$. Let $f: C\to C$ be a contraction mapping with constant $\alpha\in(0,1)$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0\in C$ arbitrarily, let the iterative sequence $\{x_n\}$ be generated by*

$$x_{n+1}=\alpha_n\gamma f(x_n)+(1-\alpha_n P)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds,$$

where $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying the following conditions:

1. $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$;
2. $\lim_{n\to\infty}\frac{|s_{n+1}-s_n|}{s_{n+1}}=0$.

Then the sequence $\{x_n\}$ converges strongly to $z\in \mathrm{F}ix(S)$, where $z=P_{\mathrm{F}ix(S)}f(z)$.

Proof. Taking $H_1 = H_2 = H$, $u_n = x_n$, and $F_1 = F_2 = 0$ in Theorem 3.1 then the conclusion of Corollary 3.2 is obtained.

Corollary 3.3 *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A: H_1 \to H_2$ be a bounded linear self adjoint operator. Assume that $F_1: C \times C \to \mathbb{R}$ and $F_2: Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in first argument. Let T be a nonexpansive mapping on C such that $Fix(T) \cap \Gamma \neq \emptyset$. Let $f: C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and P be a strongly positive bounded linear operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n); \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n P)Tu_n, \end{cases}$$

where $r_n \subset (0, \infty)$ and $\delta \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;.

(ii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;.

Then the sequence $\{x_n\}$ converges strongly to $z \in \mathrm{F}ix(T) \cap \Gamma$, where $z = P_{\mathrm{F}ix(T)\cap\Gamma}(I - P + \gamma f)z$.

Proof. Taking $T(s) := T$, a nonexpansive mapping in Theorem 3.1 then the conclusion of Corollary 3.3 is obtained.

Corollary 3.4 [26] *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty closed convex subset. Assume that $F: C \times C \to \mathbb{R}$ be the bifunctions satisfying Assumption* 2.1. *Let T be a nonexpansive mapping on C such that $Fix(T) \cap EP(F) \neq \emptyset$. Let $f: C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and P be a strongly positive bounded linear operator on H with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n P)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;. Then the sequence $\{x_n\}$ converges strongly to $z \in \mathrm{F}ix(T) \cap \mathrm{E}P(F)$, where $z = P_{\mathrm{F}ix(T)\cap\mathrm{E}P(F)}(I - P + \gamma f)z$.

Proof. Taking $F_1 = F_2 = 0$, $H_1 = H_2 = H$, $\gamma = 1$, and $P = I$ in Theorem 3.1 then the conclusion of Corollary 3.4 is obtained.

Corollary 3.5 [11] *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty closed convex subset. Assume that $F: C \times C \to \mathbb{R}$ be the bifunctions satisfying Assumption* 2.1. *Let T be a nonexpansive mapping on C such that $Fix(T) \cap EP(F) \neq \emptyset$. Let $f: C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F} x_n; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases}$$

where $r_n \subset (0, \infty)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;.

(ii) $\liminf_{n\to\infty} r_n > 0,\ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty;$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \mathrm{Fix}(T) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{Fix}(T)\cap \mathrm{EP}(F)}(I - P + \gamma f)z$.

Proof. Taking $F_1 = F_2 = F$, $H_1 = H_2 = H$, $\gamma = 1$, $P = I$ and $T(s) := T$, a nonexpansive mapping in Theorem 3.1 then the conclusion of Corollary 3.5 is obtained.

4 Numerical Example

Now, we give a numerical example which justify the main result Theorem 3.1.

Example 4.1 *Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy, \forall x, y \in \mathbb{R}$, and induced usual norm $|.|$. Let $C = [0, +\infty)$ and $Q = (-\infty, 0]$; let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined by $F_1(x, y) = x(y - x), \forall x, y \in C$ and $F_2(u, v) = (u - 2)(v - u), \forall u, v \in Q$; let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{2}x$, $A(x) = \frac{-1}{2}x$, and let, for each $x \in C$, $T(s)x = \frac{e^{-s}}{1000}x$.*

Then it is easy to prove that the bifunctions F_1 and F_2 satisfy the Assumption 2.1 and F_2 is upper semicontinuous and $T(s)x = \frac{e^{-s}}{1000}x$ is a nonexpansive semigroup and A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and $\| A \| = \| A^* \| = \frac{1}{4}$. Furthermore, take $\gamma = 1$, $\alpha_n = \frac{1}{n+1}$ and following the steps of proof of Theorem 3.1, we obtain that $\{x_n\}$ converge strongly to $w = 0 \in \mathrm{Fix}(T) \cap \Gamma$. Furthermore, the following figure shows the convergence behavior and comparison of our proposed iterative method with the iterative method of (1.6) and (1.7) with the stopping criteria is $\| x_n - p \| \leq 10^{-4}$.

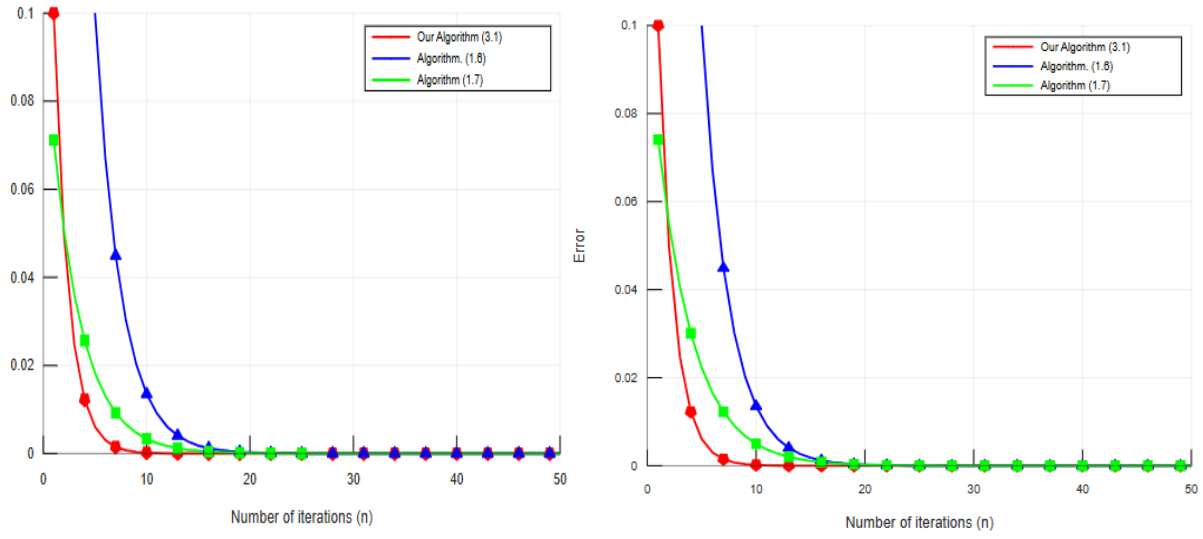


Figure: Convergence of {xn} for different initial values

Conclusion

In this paper, we have considered and studied the split equilibrium problem and fixed point problem which includes the previously known one as special cases. The aim of present work is to study the strong convergence theorem based on the modified viscosity iterative method for split equilibrium and fixed point problems for a nonexpansive semigroup. The main result extends and generalize the result of Plubtieng [27] and Cianciaruso *et al.* [28] for equilibrium problem into a split equilibrium problem and fixed point problem for a nonexpansive semigroup.

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The authors declare no conflict of interest / or state any conflict if any.

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